JOURNAL OF APPROXIMATION THEORY 8, 189–194 (1973)

# Error Bounds for a Bivariate Interpolation Scheme*

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*Communicated by John Todd*

Received August 20, 1970

DEDICATED TO PROFESSOR I. J. SCHOENBERG ON THE OCCASION OF HIS 70TH BIRTHDAY

In this note we improve the asymptotic aspect of the error bounds recently given by Hall [4] for an interpolation scheme using piecewise bivariate cubic polynomials which was suggested by Birkhoff. We make an essential use of a new "Peano kernel" type result of Bramble and Hilbert [3]. The results of this note are useful in giving sharp *a priori* error bounds for the Rayleigh–Ritz–Galerkin method used to approximate the solution of boundary value problems for elliptic partial differential equations. Throughout this note, $K$ will denote a positive constant not necessarily the same at each occurrence.

Let $R$ be any right triangular polygon in the $x - y$ plane, i.e., $R$ is the union of right triangles, $R = \bigcup_{s=1}^{k} T_s$, such that $T_s \cap T_m$, $1 \leqslant s, m \leqslant k$, is either void or a side of $T_s$ and a side of $T_m$. We are interested in interpolating smooth real-valued functions on $R$ by means of continuous, piecewise bivariate cubic polynomials $p(x, y)$, i.e., by means of functions belonging to $S_R$, where

$$S_R = \begin{cases} \{p(x, y) \mid \text{ for each } 1 \leqslant s \leqslant k, \text{ there exist real constants,} \\ a_{ij}^s \text{ such that } p(x, y) = \displaystyle\sum_{0 \leqslant i+j \leqslant 3} a_{ij}^s x^i y^j \text{ for all,} \\ (x, y) \in T_s, \text{ and } p(x, y) \in C^0(R)\}. \end{cases}$$

Moreover, if the function to be interpolated vanishes on the boundary of $R$, we may want the interpolants to do the same, i.e., to belong to $S_R{}^0 = \{p \in S_R \mid p(x, y) = 0 \text{ for all } (x, y) \text{ in the boundary of } R\}$.

* This research was supported in part by the National Science Foundation, grant no. GP 11236.

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We start by considering a single right triangle, $\Delta$, with vertices at (0, 0), $(a, 0)$, and $(0, b)$. Here $P = S_\Delta$ is the set of all restrictions to $\Delta$ of bivariate cubic polynomials. Clearly the dimension of $P$ as a vector space is 10. We define an interpolation mapping $\mathscr{I}_\Delta$ from $C^2(\Delta)$ to $P$ by

$$(D^{i,j}\mathscr{I}_\Delta f)(0, 0) = \left(\frac{\partial^{i+j}}{\partial x^i\, \partial y^j} \mathscr{I}_\Delta f\right)(0, 0) = D^{i,j}f(0, 0), \qquad 0 \leqslant i, \quad j \leqslant 1, \tag{1}$$

$$(D^{i,j}\mathscr{I}_\Delta f)(0, b) = D^{i,j}f(0, b), \qquad 0 \leqslant i + j \leqslant 1, \tag{2}$$

and

$$(D^{i,j}\mathscr{I}_\Delta f)(a, 0) = D^{i,j}f(a, 0), \qquad 0 \leqslant i + j \leqslant 1, \tag{3}$$

for all $f \in C^2(\Delta)$.

We have through [4] the following theorem.

THEOREM 1. *The interpolation mapping $\mathscr{I}_\Delta$ is well defined, i.e., $\mathscr{I}_\Delta f$ exists and is unique for all $f \in C^2(\Delta)$.*

COROLLARY. *$\mathscr{I}_\Delta \pi = \pi$ for all $p \in P$.*

Now we define a mapping $\mathscr{I}$ of $C^2(R)$ into $S_R$ as follows: If

$$f(x, y) \in C^2(R), \qquad \mathscr{I}f(x, y) \equiv s(x, y),$$

where

$$s(x, y) \equiv \mathscr{I}_{T_i}(f_i)(x, y) \qquad \text{for all} \quad (x, y) \in T_i\,, \quad 1 \leqslant i \leqslant k, \tag{4}$$

and $f_i$ denotes the restriction of $f$ to $T_i$. As corollaries of Theorem 1, we have the following theorem.

THEOREM 2. *$\mathscr{I}$ is well defined from $C^2(R)$ to $S_R$ and $\mathscr{I}(s) = s$ for all $s \in S_R$.*

*Proof.* Clearly the restriction of $\mathscr{I}(f)(x, y)$ to $T_i$ is in $P$ for all $1 \leqslant i \leqslant k$. The continuity of $\mathscr{I}(f)$ follows from the proof of Theorem 1. Q.E.D.

THEOREM 3. *$\mathscr{I}$ is well defined from*

$$C_0^2(R) \equiv \{f \in C^2(R) \mid f(x, y) = 0 \textit{ for all } (x, y) \in \partial R\}$$

*to $S_R^0$ and $\mathscr{I}(s) = s$ for all $s \in S_R^0$.*

After introducing some additional terminology, we discuss error bounds for the preceding interpolation scheme. If $j$ is a nonnegative integer and $1 \leqslant p \leqslant \infty$, we define the Sobolev norm

$$\|f\|_{W^{j,p}(R)} \equiv \left(\sum_{0 \leqslant k+l \leqslant j} \int_R |D^{k,l}f(x, y)|^p\, dx\, dy\right)^{1/p} \qquad \text{for all} \quad f \in C^\infty(R).$$

Moreover, we let $W^{j,p}(R)$ denote the completion of $C^\infty(R)$ with respect to $\|\cdot\|_{W^{j,p}(R)}$ and $W_0^{j,p}(R)$ denote the completion of $C_0^\infty(R)$ with respect to $\|\cdot\|_{W^{j,p}(R)}$.

A collection, $\mathscr{C}$, of right triangular polygons, $R$, is said to be *regular* if and only if there exists an $\epsilon > 0$ such that $\epsilon \leqslant \inf_{R\in\mathscr{C}} \inf_{1\leqslant i\leqslant k_R} h_i/H_i$, where $H_i$ and $h_i$ denote the lengths of longest and shortest sides of the triangle $T_i$, $1 \leqslant i \leqslant k_R$. We shall write $H_R \equiv \max_{1\leqslant i\leqslant k_R} H_i$.

THEOREM 4. *Let $\mathscr{C}$ be a regular collection of right triangular polygons. If $f \in W^{4,p}(R)$, (resp. $W_0^{4,p}(R)$), for $R \in \mathscr{C}$, where $p > 1$, then $\mathscr{I}f \in S_R$. (resp. $S_R{}^0$), is well defined and there exists a positive constant $K$ such that for $j = 0, 1$ and all $R \in \mathscr{C}$*

$$\|f - \mathscr{I}f\|_{W^{j,q}(R)} \leqslant K(H_R)^{4-j} \Big( \sum_{m+j=4} \| D^{m,j}f\|^p_{W^{0,p}(R)}\Big)^{1/p} \tag{5}$$

*for all $q \leqslant p$, and*

$$\|f - \mathscr{I}f\|_{W^{j,q}(R)} \leqslant K(H_R)^{4-j-(2/p)+(2/q)} \Big( \sum_{m+j=4} \| D^{m,j}f\|^p_{W^{0,p}(R)}\Big)^{1/p}, \tag{6}$$

*for all $q \geqslant p$.*

*Proof.* We consider only the case of $j = 0$, since the proof for the case of $j = 1$ is essentially identical. By the Sobolev imbedding theorem $F \in C^2(R)$, and, hence, the interpolation mapping $\mathscr{I}$ is well defined. Let $\Delta$ denote the standard right triangle with vertices at (0, 0), (1, 0), and (0, 1).

Clearly, there exists a positive constant $K$ such that

$$|f(x, y) - \mathscr{I}_\Delta f(x, y)| \leqslant K \sup_{(x,y)\in\Delta} \sum_{0\leqslant m,j\leqslant 1} | D^{m,j}f(x, y)|$$

for all $(x, y) \in \Delta$ and all $f \in C^2(\Delta)$. Moreover, since $\mathscr{I}_\Delta \pi = \pi$ for all $\pi \in \mathscr{P}$, we may apply a Peano kernel type result, corollary to Theorem 2 of [3], which states that if $(I - F)$ is a linear functional on $C^t(\Delta)$ such that there exists a positive constant $C$ such that

$$|(I - F)(u)| \leqslant C \sup_{(x,y)\in\Delta} \sum_{m+j\leqslant t} | D^{m,j}u(x, y)|$$

and $(1 - F)(p) = 0$ for all polynomials, $p(x, y)$, of degree $k > t \geqslant 0$, then for $p > 2/(R - t)$ there exists a positive constant $K$ such that

$$| F(u)| \leqslant K \sum_{m+j=k} \| D^{m,j}u \|_{W^{0,p}(\Delta)}.$$

We conclude that there exists a positive constant, again denoted by $K$, such that for all $p > 1$

$$|f(x, y) - \mathscr{I}_\Delta f(x, y)| \leqslant K \sum_{m+j=4} \| D^{m,j} f \|_{W^{0,p}(\Delta)}$$

By a standard argument, involving a change of the independent variables, cf. [1] and [3], we have, using the regularity of $\mathscr{C}$,

$$|f(x, y) - \mathscr{I} f(x, y)| \leqslant K(H_i)^{4-(2/p)} \sum_{m+j=4} \| D^{m,j} f \|_{W^{0,p}(T_i)}, \tag{7}$$

for all $(x, y) \in T_i$, $f \in W^{4,p}(R)$, $1 \leqslant i \leqslant k_R$, and all $R \in \mathscr{C}$.

To prove (5) we note that by inequality (7), if $q \leqslant p$,

$$\begin{aligned}
\| f - \mathscr{I} f \|^q_{W^{0,q}(R)} \\
&= \sum_{i=1}^{k_R} \| f - \mathscr{I} f \|^q_{W^{0,q}(T_i)} \leqslant \sum_{i=1}^{k_R} \| f - \mathscr{I} f \|^p_{W^{0,p}(T_i)} \\
&\leqslant (1/2)\, H_R{}^2 \sum_{i=1}^{k_R} \| f - \mathscr{I} f \|^p_{W^{0,\infty}(T_i)} \leqslant K(H_R)^{4p} \sum_{i=1}^{k_R} \Big( \sum_{m+j=4} \| D^{m,j} f \|_{W^{0,p}(T_i)} \Big)^p \\
&\leqslant K(H_R)^{4p} \sum_{i=1}^{k_R} \sum_{m+j=4} \| D^{m,j} f \|^p_{W^{0,p}(T_i)} = K(H_R)^{4p} \sum_{m+j=4} \| D^{m,j} f \|^p_{W^{0,p}(R)},
\end{aligned}$$

where we have used Jensen's inequality to obtain the last inequality.

To prove (6) write

$$v_i = \| f - \mathscr{I} f \|_{W^{0,q}(T_i)} \qquad \text{and} \qquad w_i = \sum_{m+j=4} \| D^{m,j} f \|_{W^{0,p}(T_i)}$$

for all $1 \leqslant i \leqslant k_R$. By (7), $v_i \leqslant K(H_i)^{4-(2/p)+(2/q)} w_i$, $1 \leqslant i \leqslant k_R$. Hence, by Jensen's and Hölder's inequalities, we have

$$\begin{aligned}
\| f - \mathscr{I} f \|_{W^{0,q}(R)} &= \Big( \sum_{i=1}^{k_R} v_i{}^q \Big)^{1/q} \leqslant K(H_R)^{4-(2/p)+(2/q)} \Big( \sum_{i=1}^{k_R} w_i{}^q \Big)^{1/q} \\
&\leqslant K(H_R)^{4-(2/p)+(2/q)} \Big( \sum_{i=1}^{k_R} w_i{}^p \Big)^{1/p} \\
&\leqslant K(H_R)^{4-(2/p)+(2/q)} \Big( \sum_{m+j=4} \| D^{m,j} f \|^p_{W^{0,p}(R)} \Big)^{1/p}
\end{aligned}$$

Q.E.D.

By making minor changes in the proof of Theorem 3, it is possible to obtain the following result.

THEOREM 5. *Let $\mathscr{C}$ be a regular collection of right triangular polygons. If $f \in W^{3,p}(R)$, (resp. $W_0^{3,p}(R)$), for all $R \in \mathscr{C}$, where $p > 2$, then $\mathscr{I}f \in S_R$, (resp. $S_R{}^0$), is well defined and there exists a positive constant, $K$, such that for $j = 0, 1$ and all $R \in \mathscr{C}$*

$$\| f - \mathscr{I}f \|_{W^{j,q}(R)} \leqslant K(H_R)^{3-j} \Big( \sum_{m+j=3} \| D^{m,j} f \|^p_{W^{0,p}(R)} \Big)^{1/p}, \tag{8}$$

*for all $q \leqslant p$, and*

$$\| f - \mathscr{I}f \|_{W^{j,q}(R)} \leqslant K(H_R)^{3-j-(2/p)+(2/q)} \Big( \sum_{m+j=3} \| D^{m,j} f \|^p_{W^{0,p}(R)} \Big)^{1/p}, \tag{9}$$

*for all $q \geqslant p$.*

We now turn to the application of Theorems 4 and 5 to obtaining error bounds for the Rayleigh–Ritz–Galerkin method for approximating the solutions of elliptic partial differential equations. In particular, we let $\bar{\Omega}$ be a closed convex polygon in the plane, $\Omega = \bar{\Omega} - \partial\bar{\Omega}$, and consider the problem of approximating the solution of

$$-D^{1,0}(p(x, y)\, D^{1,0}u) - D^{0,1}(q(x, y)\, D^{0,1}u) + r(x, y)u = f(x, y), \tag{10}$$

for all $(x, y) \in \Omega$,

$$u(x, y) = 0, \qquad \text{for all} \quad (x, y) \in \partial\bar{\Omega}, \tag{11}$$

where $p(x, y)$ and $q(x, \delta)$ are positive, real-valued, $C^1(\bar{\Omega})$ functions, $r(x, y)$ is a nonnegative, real-valued, $C(\bar{\Omega})$ function, and $f(x, y)$ is a real-valued function in $W^{0,2}(\Omega)$, by the Rayleigh–Ritz–Galerkin method. That is, if $S$ is a finite dimensional subspace of $W_0^{1,2}(\Omega)$, we must determine $u_S \in S$ such that

$$\int_\Omega p(x, y)\, D^{1,0}u_S\, D^{1,0}\varphi \, dx\, dy + \int_\Omega q(x, y)\, D^{0,1}u_S\, D^{0,1}\varphi \, dx\, dy$$
$$+ \int_\Omega r(x, y)\, u_S\varphi \, dx\, dy = \int_\Omega f(x, y)\, \varphi \, dx\, dy, \tag{12}$$

for all $\varphi \in S$.

Using the results of [2] and [5] and Theorems 4 and 5, we may establish the following error bound for the Rayleigh–Ritz–Galerkin method. The reader is referred to [5] for the precise details of the proof.

THEOREM 6. *Let $\mathscr{C}$ be a regular collection of right triangular polygonal partitions, $R$, of $\bar{\Omega}$ and for each $R \in C$, let $S_R{}^0$ denote the finite dimensional space of piecewise, bivariate cubic polynomials with respect to $R$ which vanish on the boundary of $\Omega$. Under the above hypotheses, problem* (10)–(11) *has a unique solution, $u$, $u \in W^{2,2}(\Omega)$, and if $u_R$ denotes the Rayleigh–Ritz–Galerkin approximation in $S_R{}^0$ then there exists a positive constant, $K$, such that*

$$\| u - u_R \|_{W^{j,2}(\Omega)} \leqslant K H_R^{p-j} \| u \|_{W^{p,2}(\Omega)}, \qquad 0 \leqslant j \leqslant 1, \tag{13}$$

*for all $R \in \mathscr{C}$ and all $u \in W^{p,2}(\Omega)$, where $2 \leqslant p \leqslant 4$.*

We remark that the exponent of $H$ in (13) is "best possible" for the class of solutions under consideration.

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